# Considerations on the method for calculating uncertainty in *Document CCT/08-19/rev*

The method for the calculation of $u(W_\mathrm{r})$ presented in the Document CCT/08-19/rev, it has been developed and published by D.R. White and P. Saunders in a sustained series of papers since the late 1990s [1, 2, 3, 4]. The novelty issue described in the Document as "*application of the ISO guide via interpolation theory*" is merely formal and involves "rewriting the SPRT interpolation equations in terms of a set of interpolating functions", $f_i$, meaning (C.7):

$$W_\mathrm{r}(W) = \sum_{i=1}^{N} W_{\mathrm{r},i}\ f_i(W)\,, \tag{1}$$

"which are also the fixed-point sensitivity coefficients"[1] (page 47) from the equation of error propagation[2] (C.9):

$$\mathrm{d}W_\mathrm{r} \approx -\sum_{i=2}^{N} f_i(W)\,\mathrm{d}W_i + \mathrm{d}W\,. \tag{2}$$

The authors firmly state that the intermediary values have to be determined by **interpolation between reference resistance ratios**, $W_\mathrm{r}$, the method by which equation (C.7) was derived, and not by **interpolation between the measured deviations at the fixed points** (see Note 6, page 62). It is also their assessment that one of the benefits of expressing the ITS-90 equations in the mathematical form (C.7) (equation (1) above) is the "ready identification of the fixed-point sensitivity coefficients".

Actually, neither of the two statements holds up to scrutiny. It can be easily proven that the interpolation between the measured deviations yields interpolation equations at least as advantageous as those derived by the interpolation between reference resistance ratios, including handy identification of the sensitivity coefficients $c_i \equiv \dfrac{\partial W_\mathrm{r}}{\partial W_i}$ with $-f_i$ .

For the simplicity of demonstration, let us consider **the interpolation between the measured deviations** for the water – zinc subrange. If we use the most common method of algebraic solution of a system of 2 equations with 2 unknowns (the elimination method), the resulting interpolation equation is

$$\Delta W = f_\mathrm{Sn}\Delta W_\mathrm{Sn} + f_\mathrm{Zn}\Delta W_\mathrm{Zn} = \sum_{i=2}^{3} f_i\ \Delta W_i = \sum_{i=2}^{3} f_i (W_i - W_{\mathrm{r},i}),\ \text{ where } \Delta W_i = W_i - W_{\mathrm{r},i}\,, \tag{3}$$

from which it follows:

$$W_\mathrm{r} = W - \sum_{i=2}^{3} f_i\ (W_i - W_{\mathrm{r},i}) \tag{4}$$

---

[1] In reality, $f_i$ are approximations of the sensitivity coefficients $c_i \equiv \dfrac{\partial W_\mathrm{r}}{\partial W_i}$ .

[2] The procedure specified in GUM [5] for evaluating and expressing uncertainty in measurement does not make use of the law of error propagation. The "Guide's operational approach, wherein the focus is on the observed (or estimated) value of a quantity and the observed (or estimated) variability of that value, makes any mention of error entirely unnecessary" [5]. In this respect, the approach in the Document is almost reverse.

or

$$W = W_\mathrm{r} + \sum_{i=2}^{3} f_i \,(W_i - W_{\mathrm{r},i})\,. \tag{5}$$

Equation (4) is expressed, just as (C.7), in terms of "interpolating functions" $f_i$ but, in addition to (C.7), $W_\mathrm{r}$ in (4) explicitly depends, not only implicitly, on the ratios of resistances $W_i$ (and also on $W$), which facilitates the "ready identification" of the sensitivity coefficients $c_i \equiv \dfrac{\partial W_\mathrm{r}}{\partial W_i}$ with $-f_i$. Differentiation of (4) with respect to all $W_i$ and $W$ resistance ratios leads to

$$\mathrm{d}W_\mathrm{r} = \mathrm{d}W - \sum_{i=2}^{3} f_i \,\mathrm{d}W_i - \sum_{i=2}^{3}\left(\sum_{j=2}^{3}\frac{\partial f_j}{\partial W_i}\Delta W_j\right)\mathrm{d}W_i + \sum_{i=2}^{3}\frac{\partial f_i}{\partial W}\Delta W_i \,\mathrm{d}W\,, \tag{6}$$

where the deviation functions $\Delta W_i$ have very small values (within $10^{-4}$ typically). As a result, in an approximate approach, the terms 3 and 4 in (6) can be neglected and we obtain

$$\mathrm{d}W_\mathrm{r} \approx \mathrm{d}W - \sum_{i=2}^{3} f_i \,\mathrm{d}W_i\,, \tag{7}$$

which is exactly (C.9) (Equation (2) above).

At the same time, differentiation of (C.7) (equation (1) above) - obtained by **interpolation between reference resistance ratios -** leads to the form of the equation for the propagation of error:

$$\mathrm{d}W_\mathrm{r} = \sum_{i=1}^{3}\left(\sum_{j=1}^{3}\frac{\partial f_j}{\partial W_i} W_{\mathrm{r},j}\right)\mathrm{d}W_i + \sum_{i=1}^{3}\frac{\partial f_i}{\partial W} W_{\mathrm{r},i}\,\mathrm{d}W\,, \tag{8}$$

that according to the Document is allegedly equivalent with (C.9) (equation (2) above). Actually, due to the presence of $W_{\mathrm{r},j}$ in equation (8), this equation can not be turned analytically to form (C.9), only a numerical approach can be used.

If we compare (4) and (6) – which were obtained through **interpolation among the measured deviations at the fixed points** – with (1) and (8), that were obtained through **interpolation among reference resistance ratios** – it is impossible to claim the advantage of the latter over the first ones and to demonstrate the benefit of "rewriting the SPRT interpolations" in the form C.7 as opposed to (4) or (5).

Moreover, interpolation between the measured deviations at the fixed points reveals the flexible mathematical structure of the SPRT sub-ranges, as conceived by the designers of the ITS-90. It can be easily shown that the uncertainties resulting from the calibration at the fixed points, $u(W_i)$, propagate in the same way in $u(W_\mathrm{r})$ and $u(W)$ (see the second term, $\sum_{i=2}^{3} f_i \,(W_i - W_{\mathrm{r},i})$, in equation (4), as well as in equation (5)). Thus, the mathematical foundation of ITS-90 provides two equivalent paths to evaluate the uncertainty associated with an intermediate value, either by $W_\mathrm{r}$, or by $W$.

The other elements of the interpolation model that are presented in Annex C: the differentiation of the ratio $W = R/R_{\mathrm{H20}}$ or the grouping of selected terms and the extraction of the common factor $\mathrm{d}R_{\mathrm{H2O}}$, all of them may be applied as well to the equations derived by interpolation among the measured deviations at the fixed points.

Equation (C.9) is simple and therefore looks like an advantageous way to calculate $u(W_\mathrm{r})$. But its simplicity comes with the cost of omitting the (hard to assess) covariances $u(W_i, W_j)$

among the resistance ratios at the fixed points (see #5 in [6]). Therefore, I find it highly advisable that the revision (urgently needed) of the Document CCT/08-19/rev:

**a.** either rectifies the current method for the calculation of $u(W_r)$, by taking into account the covariance terms, if a practical manner to evaluate them is found, or

**b.** replaces the current method with the models in [7, 8], so that the correlations among the $W_i$ are avoided.

Sonia Gaiță

6 January 2014

## References

[1] White D. R. (2001): *On the use of least squares and redundant fixed points with ITS-90 SPRT interpolations*, *CCT/01-08*

[2] White D. R. (2001): *The propagation of uncertainty with non-Lagrangian interpolation,* Metrologia, **38**, 63-69

[3] White D. R. (2002): *Propagating uncertainty in the SPRT subranges of ITS-90. Technical Guide 3*, Measurement Standards Laboratory

[4] White D. R. (1998): *The contribution of uncertainties in resistance measurements to uncertainties in the realization of the ITS-90*; EUROMET Workshop on Temperature, Paris, pp. 13-18

[5] JCGM 100:2008 (2008): *Evaluation of measurement data — Guide to the expression of uncertainty in measurement*

[6] Sonia Gaita (2014): Comments on CCT/08-19/rev *Uncertainties in the Realisation of the SPRT Subranges of the ITS-90*

[7] Sonia Gaita, C. Iliescu (2001): *Propagation of combined standard uncertainties evaluated in SPRT calibration according to ITS-90*, Proceedings of the International Conference of Metrology, Bucharest, Romania (ISBN 973-99385-5-8)

[8] Sonia Gaita (2001): *Sensitivity analysis of combined standard uncertainties evaluated in SPRT's calibration according to the ITS-90*; Proceedings of the International Congress "Metrology 2001", Saint-Louis, France